THE RATIONAL AND THE REAL

Numbers come in two types, with two distinct purposes. These are: the rational numbers -- used to count -- and the real numbers -- used to measure. The words “rational” and “real” provide an irresistible, and I will suggest, useful, pun.

The root word of “rational” is “ratio.” This is because the rational numbers are those that can be formed by taking ratios of the integers (the numbers 1, 2, 3 …). Fundamentally, then, the rational numbers (aka “fractions”) are based on the integers and are about counting. Counting using integers assumes that the thing being counted exists in discrete packages. We could count the number of words on this page and then ask what fraction of them contain the letter e, only because we recognize the words on this page as distinct entities, separate from each other and from the page on which they reside.

The real numbers combine the rational and the irrationals, the latter being numbers that cannot be expressed as fractions. Pythagoras (550 BC) is generally given credit for discovering that such numbers exist, when he proved that the square root of two is not rational. The number pi is perhaps the most famous irrational number but virtually all roots (square, cube, fourth, etc.) are irrational. (In fact the n^{th} root of an integer is either an integer or is irrational.) Georg Cantor, around 1900 demonstrated that the reals and the rationals are fundamentally different types of numbers -- specifically that there are more real numbers than rational numbers. This may seem strange, given that there are already an infinite number of rational numbers, but it is demonstrable that there are different orders of infinity, and that the reals belong to a larger set than do the rationals. By contrast, the rationals belong to the same order of infinity as the integers: there are no “more” fractions than there are integers.

A characteristic that separates the reals and the rationals is that rationals will, when written as decimals, always begin to repeat a fixed block of numbers, and irrational numbers will not. The simplest cases are ½, which pretty quickly begins to repeat zeros, and $\frac{1}{3}$, which immediately repeats 3’s. The decimal expansion of 3/13 is 0.230769230769 … and while this expansion goes on forever, it is unnecessary to go any further. We can see that the sequence 230769 is merely going to repeat itself *ad infinitum.* We already know everything there is to know about the fraction. For instance we can know that the 1000^{th} digit in this expansion is a 7, without having to actually write out all 1000 places. And so it is with any rational number: it is essentially *knowable*.

In contrast, the irrationals are essentially unknowable. The decimal expansion of pi (or the square root of 2, or the 7^{th} root of 12) goes on forever without ever repeating itself, without exhibiting any pattern. If one wants to know the 1000^{th} number in the expansion of pi, one must find some reliable method of calculating pi to one thousand places. But we can never know *all* about pi as a decimal. That would require the impossible task of computing an infinite number of places. This does not make pi any less of a number than the number 2: they both occupy points on a number line.

Crucial to the notion of integers and counting is that one be able to identify a “thing.” If we choose to count the glasses on the table, we must identify what is a glass and what is not. This may be problematic: is a saltshaker a “glass?” is a beer mug? This leads to the issue of sets, which is central to the point of this piece. The definition of the concept of “set” is itself difficult and I will not try to unravel it. I will define a set as, “a collection of objects defined by a rule.” Thus counting the numbers of glasses on the table presupposes that we have a rule for defining what is and is not a glass.

In any given instance, a group of people may be able to decide what elements should be included in the set, but it is crucial to see that this is a human endeavor, that we are classifying elements in the *real* world according to *rational* human thought (all puns intended and hoping to avoid the question of whether other animals also have rational faculties.). Integers are used to count "things." We can have no integers until we can decide on what constitutes a "thing" for our specific purpose.

The dictum, "You can't add apples and oranges," attests that, prior to the concept of addition (and counting is in itself an addition concept), there exists the more fundamental concept of "same or different." It is only by labeling things as "same" that we are able to enter the world of counting, of integers, and hence, the rationals.

Consider the question, "How many glasses are on the table?" Answering this requires that we apply human classifications: We need to define "glass," "the table," and "on." (This may be arduous. No less than an American President found it necessary to have the word "is" defined.) Having done that, we arrive at a perfectly precise, unambiguous answer. There is now the "set of glasses on the table," and it is perfectly well defined. This ability to classify things, to see "sameness within differences," is, I firmly believe, the fundamental, defining quality of intelligence. It is this quintessentially "rational" act that has created for humankind the world of rational numbers, where everyday arithmetic works, and where things, such as, "How many glasses are on the table," and "What is the 1000^{th} number in the decimal expansion of 3/13", are knowable. (It also clearly points out the sphere within which human intelligence is currently superior to computer ability. The difficulty in programming a machine to see "sameness within differences," means that tasks such as voice and handwriting recognition, easy for humans, are difficult for machines. We stand to learn much about the nature of human intelligence as machines get better at these tasks.)

Now consider the question, "How long is the table?" This question does not have a definitive answer; the answer depends on the measuring device being used. As one uses a progressively better measuring device, one gets closer to the truth, but never arrives. This is similar to the case of trying to compute the decimal expansion of an irrational (that is a "real") number, such as pi. You can get closer and closer, but you can't get "there." You can never know all that is to be known about the length of the table. (A solution to the problem appears to be to return to the world of rationals: the table is, after all, composed of atoms. Can't we enumerate and measure them? For several reasons the answer is, "No." First, at the table's edge, it would be impossible to decide which atoms belongs to the table and which do not, second, the fact that they are all moving would make measuring a distance impossible. Furthermore, it appears that, at the quantum level, the entire rational concept of "thing" may disappear.)

So we live in two worlds. The first is the rational world where elements can be counted and added, where statements can exactly evaluated as true or false, and where sets of objects can be unambiguously defined. But this world exists only by virtue of categories created by rational human thought, and the seemingly ubiquitous human capacity and need to classify and create sets in an attempt to understand reality.

The second world is the real world, which consists of continuums, not discrete categories. In this world no sets will precisely capture the truth. At the edges of any category lie the exceptions, the shades of gray. This is world in which no statement is ever completely true, in which one's understanding of a phenomenon is in direct relation to one's contact with it -- the equivalent of the number of decimal places to which one has computed pi.

Mathematics (until one reaches the esoteric frontiers of the subject) is the classic example of complete knowledge. But that is only true as long as mathematics deals with the discrete inventions of the human mind. Geometry provides perfect knowledge only concerning perfect, non-existent lines and circles.

So what? The central point of this argument is that any attempt to create sets that definitively classify real phenomena will be unsuccessful. There are no sets in the real world. This is not to say that creating categories is wrong or unwise, only that categories must be created with the understanding that they will fail at some point, under some conditions.

Two applications of this occur to me. The first relates to the making of laws. A law is an attempt to define a set: the set of behaviors that will be dealt with in some fashion. But any definition will fail and we must be ready to fall back on the informed judgment of involved parties. This suggests that the legislation of uniform, mandatory punishments for particular crimes is unwise.

The second application refers to attempts in science -- biology comes to mind -- to classify life into sets such as “mammal.” The category “mammal” does not exist in nature; it is merely a (probably useful) human attempt to organize reality. Arguing about whether a platypus is or is not a mammal is pretty fruitless. We can classify it however we want once realize that it isn’t “really” anything other than what it is.

Probably the ultimate usefulness of the “no-sets” hypothesis is that it provides a label for a phenomenon, since concepts are typically easier to handle once they have a name. With those with whom I have had this discussion and established the vocabulary, it is surprising how often a discussion winds up with, “Well, we’re in to the no-sets realm at this point.” This avoids tedious semantic squabbles or discussions of slippery slopes or attempts to find subtle counter-examples. Most people recognize the concept; it’s just nice to have a name for it.